DEF (Elliptic curve over a scheme S)

It is a smooth proper morphism $p: E \to S$, with a section $e: S \to E$ s.t. the pullback to any geometric fiber is an elliptic curve over $\mathrm{spec}(\bar{k})$, i.e. an integral regular scheme of dim 1, proper over k, of genus 1, with a fixed point $e_{\bar{k}}: \mathrm{Spec}\,\bar{k} \to E_{\bar{k}}$.

EXAMPLE 1

$$\begin{array}{c} \mathrm{Spec}\left(k[x,y]/y^2 = x^3 - x^2\right) \\ \big\downarrow \\ \mathrm{Spec}\,k \end{array}$$

This is the affine version, the projective version is

$$\begin{array}{c} \mathrm{Proj}\left(k[x,y,z]/y^2 z = x^3 - x^2 z\right) \\ \big\downarrow \\ \mathrm{Spec}\,k \end{array}$$

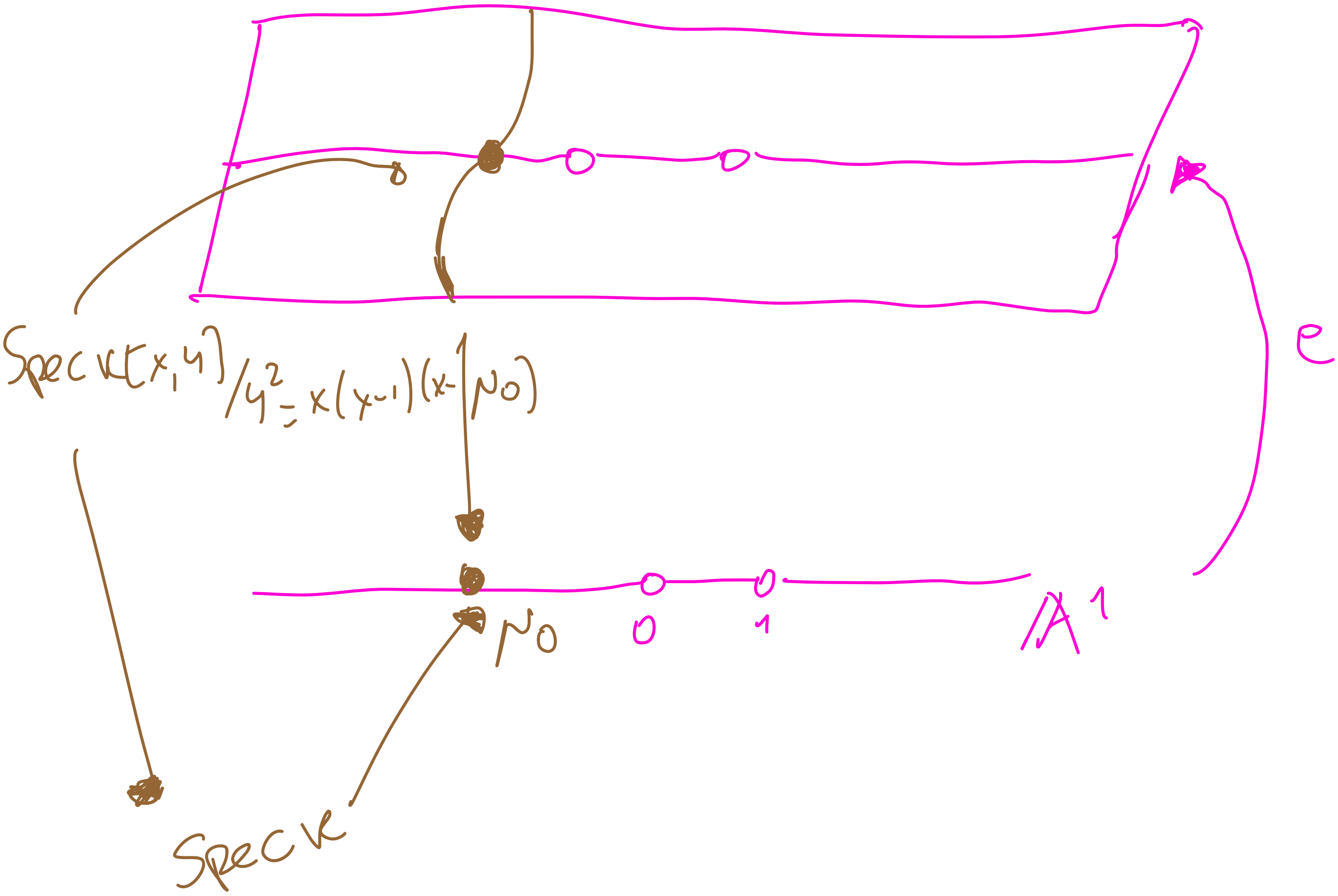
$$e: \mathrm{Spec}\,k \to \mathrm{Proj}\left(k[x,y,z]/y^2 z = x^3 - x^2 z\right)$$

$$0 \longmapsfrom (x, y-1, z)$$

$$(0:1:0)$$

EXAMPLE 2

$$\begin{array}{c} \mathrm{Spec}\left(k[x,y,\lambda]/y^2 = x(x-1)(x-\lambda)\right) \\ \big\downarrow \\ \mathbb{A}^1_\lambda - \{0,1\} = \mathrm{Spec}\,k[\lambda]_{(\lambda)(\lambda-1)} \end{array}$$



DEF

$$F : (Sch_{/k})^{op} \longrightarrow (Set)$$

$$\begin{array}{ccc} S & \longmapsto & \left\{ \begin{array}{ccc} E & \longrightarrow & S \\ \downarrow & \circlearrowleft & \downarrow \\ k & = & k \end{array} \;\middle|\; \text{families of elliptic curves} \right\} \Big/ \sim_{\text{isomorphism}} \\ \downarrow & & \\ k & & \end{array}$$

$$S_1 \longrightarrow S_2 \qquad \begin{array}{c} E_2 \\ \downarrow \\ S_2 \end{array} \longmapsto \begin{tikzcd} E_2 \times_{S_2} S_1 \arrow[r, dashed] \arrow[d] & S_1 \arrow[d, dashed] \\ S_1 \arrow[r, dashed] \arrow[ur, phantom, "\circlearrowleft"] & S_2 \end{tikzcd}$$

Objective: We would like to find a scheme M_{11} that represents F, i.e. such that

$$F(\cdot) \cong \operatorname{Hom}_{(Sch/k)}(\cdot, M_{1,1})$$

as functors.

Why is this our dream?

- They provide examples of higher dimensional algebraic varieties with a rich and interesting geometry.
- The study of moduli gives answeres to problems concerning the geometry of the objects of the family.

By Yoneda lemma:

$$\left\{ \begin{array}{c} \text{Natural transformations} \\ F(\cdot) \leftarrow \mathrm{Hom}_{Sch/k}(\cdot, M_{1,1}) \end{array} \right\} \xleftrightarrow{1:1} F(M_{1,1})$$

suppose that we have $\eta : F(\cdot) \leftarrow h_{M_{1,1}}(\cdot)$

$A \xrightarrow{\beta} B$

$$\begin{tikzcd}
? & & \varphi \arrow[ll, mapsto] \\
F(A) & & \mathrm{Hom}(A, M_{1,1}) \arrow[ll, "\eta_A"'] \\
F(B) \arrow[u, "F(\beta)"] & & \mathrm{Hom}(B, M_{1,1}) \arrow[ll, "\eta_B"] \arrow[u, "\cdot \circ \beta"']
\end{tikzcd}$$

In particular if I take $B = M_{1,1}$

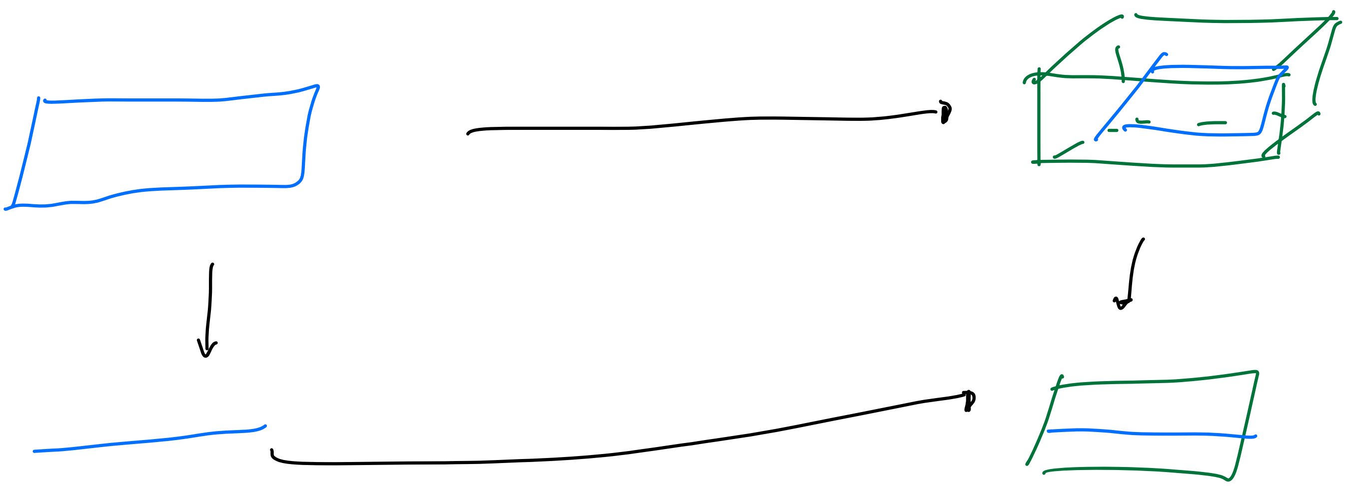
$A \xrightarrow{\varphi} M_{1,1}$

$$\begin{tikzcd}
a' & & & \varphi \arrow[lll, mapsto] \\
F(A) & & \mathrm{Hom}(A, M_{1,1}) \arrow[ll] & \\
F(M_{1,1}) \arrow[u, "F(\varphi)"] & & \mathrm{Hom}(M_{1,1}, M_{1,1}) \arrow[ll] \arrow[u, "\circ \varphi"'] &
\end{tikzcd}$$

$\xi \leftarrow\!\shortmid \mathrm{id}$

So consider the element $F(M_{11}) \ni \xi$ that is the image of $\mathrm{id} \in \mathrm{Hom}(M_{11}, M_{11})$ via the natural isomorphism η. This element ξ is an elliptic curve.

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\rho"'] \arrow[dr, phantom, "\square"] & \xi \arrow[d] \\ S \arrow[r, "\exists! \varphi"'] & M_{11} \end{tikzcd}$$



A GOOD ATTEMPT:

FACT 1:

In char $k \neq 2,3$ every elliptic curve is isomorphic over $\bar{k}$ to

$E_\lambda: \quad y^2 = x(x-1)(x-\lambda) \qquad \lambda \neq 0,1.$

$$J(E_\lambda) = 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2} \qquad \begin{aligned} \overline{k} \setminus \{0,1\} &\xrightarrow{\varphi} \overline{k} \\ \lambda &\longmapsto J(E_\lambda) \end{aligned}$$

φ is surjective and $6:1$, except in $J=0$ and $J=1728$ where it is $2:1$ and $3:1$.

Two elliptic curves are isomorphic $\Longleftrightarrow$ they have the same J-invariant

FACT 2:

Theorem

Let $E \to S$ elliptic curve. $\exists \{U_i = \operatorname{Spec} A_i\}_i$ zariski opens for S such that

$$\mathbb{P}^2_{A_i} / (y^2 = X^3 + AX + B) \cong \begin{tikzcd} \overbrace{E \times_S \operatorname{Spec} A_i}^{E_i} \arrow[r] \arrow[d] & E \arrow[d] \\ \operatorname{Spec} A_i \arrow[r] & S \end{tikzcd}$$

The J_i- invariants of $E_i \to \operatorname{Spec} A_i$, which are elements of $\Gamma(\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i} = A_i)$ coincides on the overlaps and therefore they lift to a $J \in \Gamma(S, \mathcal{O}_S)$.

So our best candidate for M_{11} is

$$\begin{tikzcd}
E \arrow[r, dashed, "\text{HOPE THIS IS PULLBACK}"] \arrow[d] & \xi \arrow[d] \\
S \arrow[r, "j"'] & \mathbb{A}^1_j
\end{tikzcd}$$

$$\mathcal{O}_S \longleftarrow k[j]$$

$$j(E) \longleftarrow\!\shortmid\ j$$

This doesn't work.

$$\begin{tikzcd}
E_1 = \operatorname{Spec} k[x,y,t]/_{y^2=x^3-t} \arrow[r] \arrow[d] & \xi \arrow[d] \\
S = \mathbb{A}^1_t \setminus \{0\} = \operatorname{Spec} k[t] \setminus \{0\} \arrow[r] & \mathbb{A}^1
\end{tikzcd}$$

$$j(E) = 0$$

$$\begin{tikzcd}
E_2 = \operatorname{Spec} k[x,y,t]/_{y^2=x^3-1} = \operatorname{Spec} k[x,y]/_{y^2=x^3-1} \times \mathbb{A}^1_t \setminus \{0\} \arrow[d] \\
S = \mathbb{A}^1_t \setminus \{0\}
\end{tikzcd}$$

Contradiction.

Bad NEWS: Such a scheme $M_{1,1}$ doesn't exist.

SLOGAN: The presence of nontrivial automorphisms prevent the moduli problem from having a fine moduli space.

Intuition: The curves $y^2 = x^3 - t$ and $y^2 = x^3 - 1$ in $\mathbb{P}^2_{\mathbb{C}(t)}$ are not isomorphic but they are on $\mathbb{P}^2_{\mathbb{C}(t^{1/6})}$.

$$\begin{cases} x = u^2 x' \\ y = u^3 y' \end{cases} \quad u \text{ invertible}$$

So suppose $y'^2 = x'^3 - t$ and we want $y^2 = x^3 - 1$. Then

$$(u^3 y')^2 = (u^2 x')^3 - 1 \quad \text{i.e.}$$

$$u^6 (x'^3 - t) = u^6 x'^3 - 1 \quad \text{i.e.}$$

$$u^6 t = 1 \quad \rightsquigarrow \quad u = t^{-1/6}$$

$$
\begin{tikzcd}
& \widetilde{E}_2 \arrow[rr] \arrow[ddl] & & E_2 \arrow[ddl] \\
\widetilde{E}_1 \arrow[ur, "\cong"] \arrow[rr] \arrow[d] & & E_1 \arrow[d] & \\
\operatorname{Spec}\left(\mathbb{C}[t,t^{-1},y]/_{y^6-t}\right) \arrow[rr] & & \operatorname{Spec}\left(\mathbb{C}[t,t^{-1}]\right) &
\end{tikzcd}
$$

$$y \xleftarrow[\text{étale}]{} t$$

Two approaches to circumvent the problem:

1. Restrict the class of objects of our families to eliminate automorphisms, i.e. rigidify the moduli problem.

   Machinery of Geometric Invariant Theory (GIT) (Mumford)

2. Record the info about automorphism enlarging the category of schemes to ensure representability of the moduli functor.

Solution: We wanted a scheme $M_{1,1}$ such that we could identify the functor F with $M_{1,1}$.

This scheme doesn't exists. So "we identify F with F itself and we call it $\mathcal{M}_{1,1}$.

So for every $T \in (Sch)$ $\mathcal{M}_{1,1}(T)$ is the groupoid of elliptic curves over T.

Trivial: $\mathcal{M}_{1,1}$ is category fibered in groupoids.

THEOREM

$\mathcal{M}_{1,1}$ is a stack for the FPQC topology.

$\boxed{\text{PROOF}}$ [Algebraic Spaces and Stacks, M. Olsson] = [ASS] ch 13.1 $\square$

What does it mean? That:

$\forall$ scheme S $\quad \forall \{U_i \to S\}$ covering

$\mathcal{M}(S) \longrightarrow \mathcal{M}(\{U_i \to S\})$ is equivalence of category.

$$\begin{tikzcd} E \arrow[d] \\ S \end{tikzcd} \longmapsto \begin{tikzcd} E_i \arrow[r] \arrow[d] & E \arrow[d] \\ U_i \arrow[r] & S \end{tikzcd}$$

$$\begin{tikzcd} E_i \times_{U_i} U_{i,j} \arrow[r, "\psi_{ij}"] \arrow[d] & E_j \times_{U_j} U_{i,j} \arrow[dl] \\ U_{i,j} & \end{tikzcd}$$

+ cocycle conditions

+ morphisms.

Why this is the solution to our problems?

I wanted for each

$$\begin{tikzcd} E \arrow[r, dashed] \arrow[d, "h"'] \arrow[dr, phantom, "\square"] & \xi \arrow[d] \\ S \arrow[r, "\exists! \psi"'] & \mathcal{M} \end{tikzcd}$$

So fix $\begin{matrix} E \\ h \downarrow \\ S \end{matrix}$. Who is ψ? We are looking for

a morphism $S \xrightarrow{\psi} \mathcal{M}$. Here we are identifying the scheme S with the stack $\mathrm{Hom}_{(Sch)}(\bullet, S)$.

So we can think to ψ as a natural transformation

$\forall T$: $\quad S(T) = \mathrm{Hom}(T, S) \xrightarrow{\eta_T} \mathcal{M}(T)$.

Again we apply Yoneda and $\mathrm{Nat}(S, \mathcal{M}) \overset{1:1}{\leftrightarrow} \mathcal{M}(S)$.

So given $\begin{array}{c} E \\ \eta \downarrow \\ S \end{array}$ the ψ corresponds to η. $\quad \psi \longleftarrow \begin{array}{c} E \\ \downarrow \eta \\ S \end{array}$

Remark: These are not functors but pseudofunctors.

But Yoneda works in this setting too.

$\mathcal{M}$ is a "functor". Can we do geometry?

DEF

A stack $\mathcal{X} \to S$ is said to be an $\overbrace{\text{algebraic}}^{\text{(Artin)}}$ stack if

(i) The diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable, i.e $\forall$ scheme U $\forall \eta$ the pullback:

$$\begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d, "\eta"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

the pullback is an "algebraic space" (let assume a scheme).

(ii) $\exists$ a smooth surjective morphism $\pi : X \to \mathcal{X}$ with X a scheme.

Observation: What does smooth and surjective mean?

Smooth is said to be a "stable" property cause given $X \xrightarrow{f} Y$ between schemes and a covering $\{ Y_i \xrightarrow{\varphi_i} Y \}$ f is smooth $\iff$ $\varphi_i^* f$ is smooth $\forall i$.

If a morphism is representable (like π) we say it has a stable property P (like smooth) if $\forall$ scheme the pullback has this property.

In this case

$$\begin{tikzcd} V \arrow[r, "\text{smooth}"] \arrow[d] & U \arrow[d, "f"] \\ X \arrow[r, "\pi"'] & \mathcal{X} \end{tikzcd} \quad \forall f$$

<u>Remark</u>: π is representable because the fiber product of

$$\begin{tikzcd} & U \arrow[d, "f"] \\ X \arrow[r, "\pi"'] & \mathcal{X} \end{tikzcd}$$

is the same of

$$\begin{tikzcd} & X \times U \arrow[d, "(\pi, f)"] \\ \mathcal{X} \arrow[r, "\Delta"'] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

DEF An Artin Stack is said to be Deligne-Mumford if π is étale.

TORSORS AND QUOTIENT STACKS

DEF

with some hypotesys like smooth and affine

X a scheme and G groupscheme.

A G-fibration over X is a scheme ξ, with an action

$\rho : G \times \xi \to \xi$ and a G-invariant morphism $\xi \xrightarrow{\pi} X$.

Morphism defined in the obvious way.

A G-fibration is called trivial if it is isomorphic to the G-fibration $pr_2 : G \times X \to X$ where the action obvious.

DEF (Principal G-Bundle)

A principal G-bundle with respect to the fpqc (resp fppf, étale, zariski) topology is a G-fibration which is locally trivial in the fpcq (resp fppf, étale, zariski) topology. This means $\forall x \in X$ $\exists x \in U \subseteq X$ and a fpcq (resp fppf, étale, zariski) cover $\{U_i \xrightarrow{\varphi_i} U\}$ such that $\varphi_i^*(\xi|_U) \cong U_i \times G$.

EXAMPLE

X scheme, G linear algebraic group acting on X via $\rho : X \times G \to X$. Assume the action of ρ is free. Then X/G exists as an algebraic space and the quotient morphism $\tau : X \to X/G$ is a principal G-bundle.

Consider $U \xrightarrow{\beta} X/G$

$$\begin{tikzcd} U \times_{X/G} X = \xi \arrow[r, "\alpha"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\tau"] \\ U \arrow[r, "\beta"'] & X/G \end{tikzcd}$$

$\xi \xrightarrow{\pi} U$ is a principal G-bundle too!

Moreover α is G-equivariant.

What if X/G doesn't exist?

DEF (Quotient stack $[X/G]$)

$$[X/G] : (Sch)^{op} \to (Grpds)$$

$$U \longmapsto \begin{tikzcd} \xi \arrow[r, "\alpha"] \arrow[d, "\pi"'] & X \\ U \end{tikzcd}$$

If $X = \operatorname{Spec} k$ with trivial action of the group scheme $G \to \operatorname{Spec} k$ Then it is $[\operatorname{Spec} k/G]$ also denoted as BG and called classifying stack.

THEOREM

$[X/G]$ is an algebraic stack.

$\boxed{\text{PROOF}}$ [ASS] EXAMPLE 8.1.12. $\square$

The atlas is

$$X \xrightarrow{\pi} [X/G]$$

$\forall\, U \xrightarrow{h} [X/G]$ which corresponds to $\begin{matrix} \xi \\ \downarrow \\ U \end{matrix}$

we have

$$\begin{tikzcd}
X \times_{[X/G]} U \cong \xi \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d, "h"] \\
X \arrow[r, "\pi"'] & {[X/G]}
\end{tikzcd}$$

The universal torsor.

In particular

$$\begin{tikzcd}
\xi \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\
X \arrow[r, "\pi"'] & {[X/G]}
\end{tikzcd}$$

EXAMPLE

If I consider $\mathcal{M}_{1,1}$ over $\operatorname{Spec} k$ char $k \neq 2,3$ then

$$\mathcal{M}_{1,1} \cong \left[{}^{U}/_{G} \right]$$ where

$$U = \operatorname{Spec}\left(k[A, B, \tfrac{1}{\Delta(A,B)}]\right) \qquad \Delta = -16(4A^3+27B^2)$$

$$G = \mathbb{G}_m = \operatorname{Spec} k[u^{\pm 1}]$$

In general $\quad y^2 = x^3 + Ax + B$

$$x \longmapsto u^2 x' \qquad \text{induces}$$

$$y \longmapsto u^3 y' \qquad \Longrightarrow \qquad u^4 A' = A, \quad u^6 B' = B, \quad u^{12}\Delta' = \Delta$$

$$\begin{tikzcd}
\overbrace{\operatorname{Spec} k[x,y]/_{y^2=x^3+Ax+B}}^{\text{universal elliptic curve.}} \arrow[r] \arrow[d] & U \arrow[d] \\
U \arrow[r] & {[U/G]}
\end{tikzcd}$$

We also have a morphism

$$[U/G] \xrightarrow{J} \mathbb{A}^1$$

and it is its coarse moduli space,

i.e. it is a scheme s.t.

(i) $\forall g : \overset{\mathcal{M}_{1,1}}{[U/G]} \longrightarrow Z$ with Z scheme (algebraic space) then $\exists!\ \varphi$ s.t.

$$\begin{tikzcd}
\mathcal{M}_{1,1} = [U/G] \arrow[r, "J"] \arrow[dr, "g"'] & \mathbb{A}^1 \arrow[d, dashed, "\varphi"] \arrow[dl, phantom, "\circlearrowleft"] \\
& Z
\end{tikzcd}$$

(ii) $\forall\ \Omega$ alg closed field $k \subseteq \Omega$

$$\underbrace{\left\{ [U/G](\Omega) \right\} /\sim_{\text{isom}}}_{\text{isomorphism classes of elliptic curves over } \Omega} \xleftrightarrow{1:1} \underbrace{\mathbb{A}^1(\Omega)}_{\Omega\text{-points}}$$