FIBERED CAT. + STACKS

- Cartesian arrows and fibered categories
- Concretely: $Arr(\mathcal{C})$
- fibers, ~~cleavage, pseudo-funtors~~
  - fibered in sets $\Leftrightarrow$ functor
- fibered in groupoids

- examples
  - h_X
  - Arrow category (generalizes pullback)
  - $Sh/\mathcal{C}$ (the world makes sense)

- $Arr(Top)$ as motivation for stacks
- Descent data, $F(U) \to F(\{U_i \to U\})$, effective data
- ~~equivalent formulations & sieves (brief)~~
- Definition of prestack & stack
- (For fibered in sets it is the same as separated presheaf and sheaf)
- Hom-criterion for prestack
  - $Sh/\mathcal{C}$ is a stack over $\mathcal{C}$
- brief mention of functorial properties

Def $\mathcal{F}$ over $\mathcal{C}$ if we fix $P_{\mathcal{F}} : \mathcal{F} \to \mathcal{C}$ functor

Def $\phi : \xi \to \eta$ in $\mathcal{F}$ is <u>cartesian</u> if $\forall \psi, h$ s.t.

If $\xi \to \eta$ cartesian, $P_{\mathcal{F}}(\xi \to \eta) = u \to v$ then ξ is the <u>pullback</u> of η to u

$$\begin{tikzcd}
\zeta \arrow[r, dashed, "\exists!"] \arrow[rr, bend right, "\psi"'] \arrow[d, mapsto] & \xi \arrow[r] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\
P_{\mathcal{F}}\zeta \arrow[r, "h"] \arrow[rr, bend right, "P_{\mathcal{F}}\psi"'] & P_{\mathcal{F}}\xi \arrow[r, mapsto] & P_{\mathcal{F}}\eta
\end{tikzcd}$$

Facts

(i) cart. $\circ$ cert = cart

(ii)
$$\begin{tikzcd}
\zeta \arrow[r, "a"] \arrow[dr, "b"'] & \eta \arrow[d, "\text{cart}"] \\
& \xi
\end{tikzcd}$$
then a cart $\Leftrightarrow$ b cart

(iii) If $P_{\mathcal{F}}\phi$ is iso., ϕ cart. iff ϕ iso.

(iv) $\mathcal{F} \xrightarrow{F} \mathcal{G} \xrightarrow{P_{\mathcal{G}}} \mathcal{C}$

$\phi \longmapsto F\phi \longmapsto P_{\mathcal{G}}F\phi$

cart & cart $\Downarrow$ cart

Def $\mathcal{F}$ <u>fibered over</u> $\mathcal{C}$ if over $\mathcal{C}$ and $\forall f : u \to v$ in $\mathcal{C}$, $\forall \eta \in \mathcal{F}$ s.t. $P_{\mathcal{F}}\eta = v$, there is $\phi : \xi \to \eta$ cartesian s.t. $P_{\mathcal{F}}\phi = f$

(all pullbacks exist)

If $\mathcal{F}, \mathcal{G}$ over $\mathcal{C}$, $F : \mathcal{F} \to \mathcal{G}$ is a functor s.t.

$P_{\mathcal{G}} \circ F = P_{\mathcal{F}}$, $\forall \phi$ cartesian, $F(\phi)$ cartesian

Def $\begin{matrix}\mathcal{F}\\ \downarrow\\ \mathcal{C}\end{matrix}$ $U \in \mathcal{C}$, the fiber of $\mathcal{F}$ over U ($\mathcal{F}(U)$) is the subcategory of $\mathcal{F}$ where $\xi \in \mathcal{F}(U)$ when $p_{\mathcal{F}}\xi = U$ and the arrows ϕ are those s.t. $p_{\mathcal{F}}\phi = id_U$

Rem $F: \mathcal{F} \to \mathcal{G}$ morphism (over $\mathcal{C}$), $U \in \mathcal{C}$ then $F_U : \mathcal{F}(U) \to \mathcal{G}(U)$ makes sense

LOOK AT THE NOTATION! This is like evaluating a sheaf at an open set and looking at the data of a sheaf morphism on an open set respectively!!!

Rem The definition technically makes sense even for $\mathcal{F}$ over $\mathcal{C}$ not fibered, but the problem is that $U \cong V$ in $\mathcal{C}$ $\not\Rightarrow (\mathcal{F}(U) \neq \emptyset \Rightarrow \mathcal{F}(V) \neq \emptyset)$ so it would behave HORRIBLY with a topology on $\mathcal{C}$

This problem is avoided for $\mathcal{F}$ fibered over $\mathcal{C}$ because if $f: U \to V$ in $\mathcal{C}$ and η over V in $\mathcal{F}$ then $\exists\ \phi_\eta : f^*\eta \to \eta$ cartesian over f.

Now however, we find a uniqueness problem for $f^*\eta$

Def A cleavage for $\mathcal{F}$ fib. $/\mathcal{C}$ is a class K of cartesian arrows in $\mathcal{F}$ s.t. $\forall f: U \to V$ in $\mathcal{C}$ and all $\eta \in \mathcal{F}$ over V, $\exists! \phi \in \mathcal{F}$ s.t. $\phi : f^*\eta \to \eta$ over f.

Having chosen a cleavage we can define unambiguously

$$\bullet^*: \mathcal{C} \longrightarrow Cat$$
$$U \longmapsto \mathcal{F}(U)$$
$$f: U \to V \longmapsto f^*: \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$$
$$\eta \longmapsto f^*\eta$$
$$\beta: \eta \longrightarrow \eta' \longmapsto f^*\beta: f^*\eta \longrightarrow f^*\eta'$$

ok because of cleavage

It looks like we have $\bullet^*: \mathcal{C} \longrightarrow Cat$ funtor, but there is no guarantee that

$$id_U^* = id_{\mathcal{F}(U)} \quad \text{nor that} \quad f^* g^* \xi = (gf)^* \xi$$

What is true is that they are naturally iso.

$\bullet^*$ is a <u>pseudo-functor</u>: Φ pseudo-funct on $\mathcal{C}$ if

(i) $\forall\, U \in \mathcal{C}$, $\Phi(U) \in Cat$

(ii) $\forall\, f: U \to V$, $f^*: \Phi V \to \Phi U$ functor

(iii) $\forall\, U \in \mathcal{C}$, $\varepsilon_U: (id_U)^* \xrightarrow{\sim} id_{\Phi U}$ natural iso.

(iv) $\forall\, U \xrightarrow{f} V \xrightarrow{g} W$, $\alpha_{f,g}: f^* g^* \xrightarrow{\sim} (gf)^*$ natural iso.

(v) $f: U \to V$ in $\mathcal{C}$, $\eta \in \Phi V$, $\alpha_{id_U, f}(\eta) = \varepsilon_U(f^*\eta)$

$\alpha_{f, id_U}(\eta) = f^* \varepsilon_V(\eta)$

(vi) whatever associativity should be

Note A pseudo-functor gives a fibered cat + cleavage

$\Phi: \mathcal{C}^{op} \to Cat \rightsquigarrow obj\,\mathcal{F} = \{(U, \xi) \mid \xi \in \Phi U\}$, $Mor\,\mathcal{F} = \left\{(\xi, a) \,\middle|\, \begin{array}{l} f: U \to V \\ a: \xi \to \Phi f^*(\eta) \end{array}\right\}$

"A cleavage is a choice of pullbacks"

Def $\mathcal{F}$ fib/$\mathcal{C}$ is fibered in sets if $\mathcal{F}(U)$ is a set

Prop $\mathcal{F}$ fib. in sets $\iff$ $\forall \eta \in \mathcal{F}$, $\forall f: U \to p_{\mathcal{F}}\eta$
$\exists! \, \phi: \xi \to \eta$ over f

Pf [Vistoli 3.25] $\square$

Prop $(\text{fib}/\mathcal{C} \text{ in sets}) \overset{\text{equiv.}}{\cong} \text{Fun}(\mathcal{C}^{op}, \text{set})$

Pf [Vistoli 3.26]

Rem
$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "\text{Yoneda}"] \arrow[dr, "\sim"'] & \text{Fun}(\mathcal{C}^{op}, \text{set}) \arrow[r, hook] \arrow[dr, "\sim"'] & \text{Fib}/\mathcal{C} \\ & \text{representable} \arrow[u, hook] & (\text{fib}/\mathcal{C} \text{ in sets}) \arrow[u, hook] \end{tikzcd}$$

Def $\mathcal{F}$ is fibered in groupoids over $\mathcal{C}$ if fibered over $\mathcal{C}$ via $p_{\mathcal{F}}$ and $\mathcal{F}(U)$ groupoid $\forall U \in \mathcal{C}$

Prop $\mathcal{F}$ over $\mathcal{C}$ is fibered in groupoids iff
(i) Every arrow in $\mathcal{F}$ is cartesian
(ii) If $\eta \in \mathcal{F}$, $f: U \to p_{\mathcal{F}}\eta$ then $\exists \, \phi: \xi \to \eta$ s.t. $p_{\mathcal{F}}\phi = f$

Pf $\boxed{\Leftarrow}$ $\mathcal{F}$ fibered over $\mathcal{C}$; ϕ arrow exists by (ii) and it is cartesian by (i)

If $\begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ U \arrow[r, "id"] & U \end{tikzcd}$ (i.e. $\phi \in \mathcal{F}(U)$), since by (i) it is cartesian

$$\begin{tikzcd}
\eta \arrow[r, dashed, "\exists\psi"] \arrow[rr, bend right] \arrow[d, mapsto] & \xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\
U \arrow[r, "id_U"] \arrow[rr, bend right, "id_U"'] & U \arrow[r, "id_U"] & U
\end{tikzcd}$$

so all elts of $\mathcal{F}(U)$ have a right inverse in $\mathcal{F}(U)$.

$\phi\psi = id_\eta$, $\psi\chi = id_\xi$

because $\chi = \phi\psi\chi = \phi\, id_\xi = \phi$

$\rightsquigarrow \mathcal{F}(U)$ groupoid.

$\boxed{\Rightarrow}$ (ii) ok because $\mathcal{F}$ fibered over $\mathcal{C}$.

Let $\phi: \xi \to \eta$ be an arrow in $\mathcal{F}$ and let $f = p_{\mathcal{F}}\phi = U \to V$.

Choose $\phi': \xi' \to \eta$ pullback of η to U.

Since ϕ' cartesian $\exists \alpha: \xi \to \xi' \in \mathcal{F}(U)$ s.t. $\phi' \circ \alpha = \phi$

$$\begin{tikzcd}
\xi \arrow[r, dashed, "\exists\alpha"] \arrow[rr, bend right, "\phi"] \arrow[d, mapsto] & \xi' \arrow[r, "\phi'"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\
U \arrow[r, "id_U"] \arrow[rr, bend right] & U \arrow[r] & V
\end{tikzcd}$$

since $\mathcal{F}(U)$ groupoid α iso.

thus ϕ' cartesian $\Rightarrow \phi$ cartesian.

$\square$

EXAMPLES

Ex Let $\mathcal{C}$ be a cat. with fibered products

$$\mathrm{Arr}(\mathcal{C}) = \left\{ \begin{matrix} \text{arrows in } \mathcal{C} \\ \text{commutative squares} \end{matrix} \right\}$$

$$p : \mathrm{Arr}(\mathcal{C}) \longrightarrow \mathcal{C}$$
$$f : S \to U \longmapsto U$$

$$\begin{tikzcd} S \arrow[r] \arrow[d, "f"'] & T \arrow[d, "g"] \\ U \arrow[r] & V \end{tikzcd} \longmapsto U \to V$$

$\mathrm{Arr}(\mathcal{C})$ is fibered over $\mathcal{C}$: cartesian arrows ARE cartesian squares and $\mathcal{C}$ having fibered products yields the condition for fib.cat.

Pf (cart. arrow $\Rightarrow$ cart. square)

$$\begin{tikzcd} A \arrow[r, dashed] \arrow[d] \arrow[dr] \arrow[rr, bend left] & S \arrow[r] \arrow[d] & T \arrow[d] \\ A \arrow[r] & U \arrow[r] & V \end{tikzcd}$$

(cart square $\Rightarrow$ cart. arrow)

$$\begin{tikzcd} A \arrow[r, dashed] \arrow[d] \arrow[dr] \arrow[rr, bend left] & S \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d] \\ W \arrow[r] \arrow[rr, bend right] & U \arrow[r] & V \end{tikzcd}$$

$\square$

Rem $\mathrm{Arr}(\mathcal{C})(U) = \mathcal{C}/U$

Ex Let $\mathcal{C}$ be a site with topology $\mathcal{T}$

$$Sh : \mathcal{C}^{op} \longrightarrow \mathrm{Cat}$$
$$X \longmapsto ShX = \text{sheaves in } \mathcal{C}/X$$

w.r.t. comma top.

(the thing you'd expect ShX to be)

$$f : X \to Y \longmapsto f^* : ShY \longrightarrow ShX$$
$$F \longmapsto \left(U_{\to X} \mapsto F(U \to X \xrightarrow{f} Y) \right)$$

$$\begin{tikzcd}[column sep=small] U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd} \longmapsto \begin{tikzcd}[column sep=small] FU \arrow[rr] \arrow[dr] & & FV \arrow[dl] \\ & FX \arrow[d] & \\ & FY & \end{tikzcd}$$

$$F \xrightarrow{\phi} G \longmapsto f^*F \xrightarrow{f^*\phi} f^*G$$

This is a pseudo-functor, it amounts to showing that

$$Sh/\mathcal{C} \longrightarrow \mathcal{C} \quad \text{is fibered}$$

$$\begin{bmatrix} X \to Y \\ F \to f^*G \end{bmatrix}_{\mathcal{C}} \qquad (X,F) \longmapsto X$$

$$(X,F) \to (Y,G) \longmapsto X \to Y$$

EX

$$X \rightsquigarrow \quad h_X : \mathcal{C}^{op} \to Set \quad \rightsquigarrow \quad \begin{array}{c} h_X \\ \downarrow \\ \mathcal{C} \end{array}$$

$$Y \mapsto Hom(Y,X)$$

$$Y \xrightarrow{f} Z \mapsto \begin{array}{c} Hom(Z,X) \\ \downarrow \circ f \\ Hom(Y,X) \end{array}$$

$$h_X: \quad Y \to X \longmapsto Y$$

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & Z \arrow[d] \\ & X \end{tikzcd} \longmapsto Y \to Z$$

Rem h_X is fib. in groupoids:

$h_X(Y) = Hom(Y,X)$ is a set, thus it is a groupoid

(fiber over Y)

(Set = small cat. where only arrows are identities)

DESCENT

Motivating example $p: (\mathrm{Cont}) \longrightarrow (\mathrm{Top})$ is an arrow cat so fibered.
$f: X \to Y \longmapsto Y$

Given $f: X \to U$ & $g: Y \to U$ we want to build
$\phi: X \to Y \in (\mathrm{Cont})(U) = (\mathrm{Top}/U)$ s.t. $p\phi = \mathrm{id}_U$

Suppose we have $\{U_i\}$ open cover of U, $\phi_i : f^{-1}U_i \to g^{-1}U_i$
continuous & over U_i, $\phi_i|_{f^{-1}(U_{ij})} = \phi_j|_{f^{-1}(U_{ij})}$
Then $\exists!\ \phi$ s.t. $\phi|_{U_i} = \phi_i$

Concretely, let $f: V \to U$, $X \to U \in (\mathrm{Cont})(U) = (\mathrm{Top}/U)$,
then $V \times_U X \to V$ pullback of X along f.
This yields a functor $f^*: (\mathrm{Cont})(U) \to (\mathrm{Cont})(V)$

$$X \to U \longmapsto X \times_U V \to V$$
$$\phi: X \to Y \text{ over } U \longmapsto f^*\phi = \mathrm{id}_V \times_U f$$

Suppose $X, Y \in (\mathrm{Top}/S)$. Consider $\underline{\mathrm{Hom}}_S(X,Y): (\mathrm{Top}/S) \to (\mathrm{Set})$

$$u: U \to S \longmapsto \mathrm{Hom}_U(u^*X, u^*Y) = \mathrm{Hom}_U(U \times_S X, U \times_S Y)$$
$$f: U \to V \text{ over } S \longmapsto \left(\begin{array}{c} \phi: U \times_S X \to U \times_S Y \\ \downarrow \\ f^*\phi: V \times_S X \to V \times_S Y \end{array}\right)$$

The gluing condition
becomes: $\underline{\mathrm{Hom}}_S(X,Y): (\mathrm{Top}/S)^{op} \to (\mathrm{Set})$
is a sheaf for (Top) (as a site)

We can also glue topological spaces with cocycle conditions:

Prop $\mathcal{U} \in \mathrm{Top}$, $\{\mathcal{U}_i\}$ open cover, set $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j$
$\mathcal{U}_{ijk} = \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$

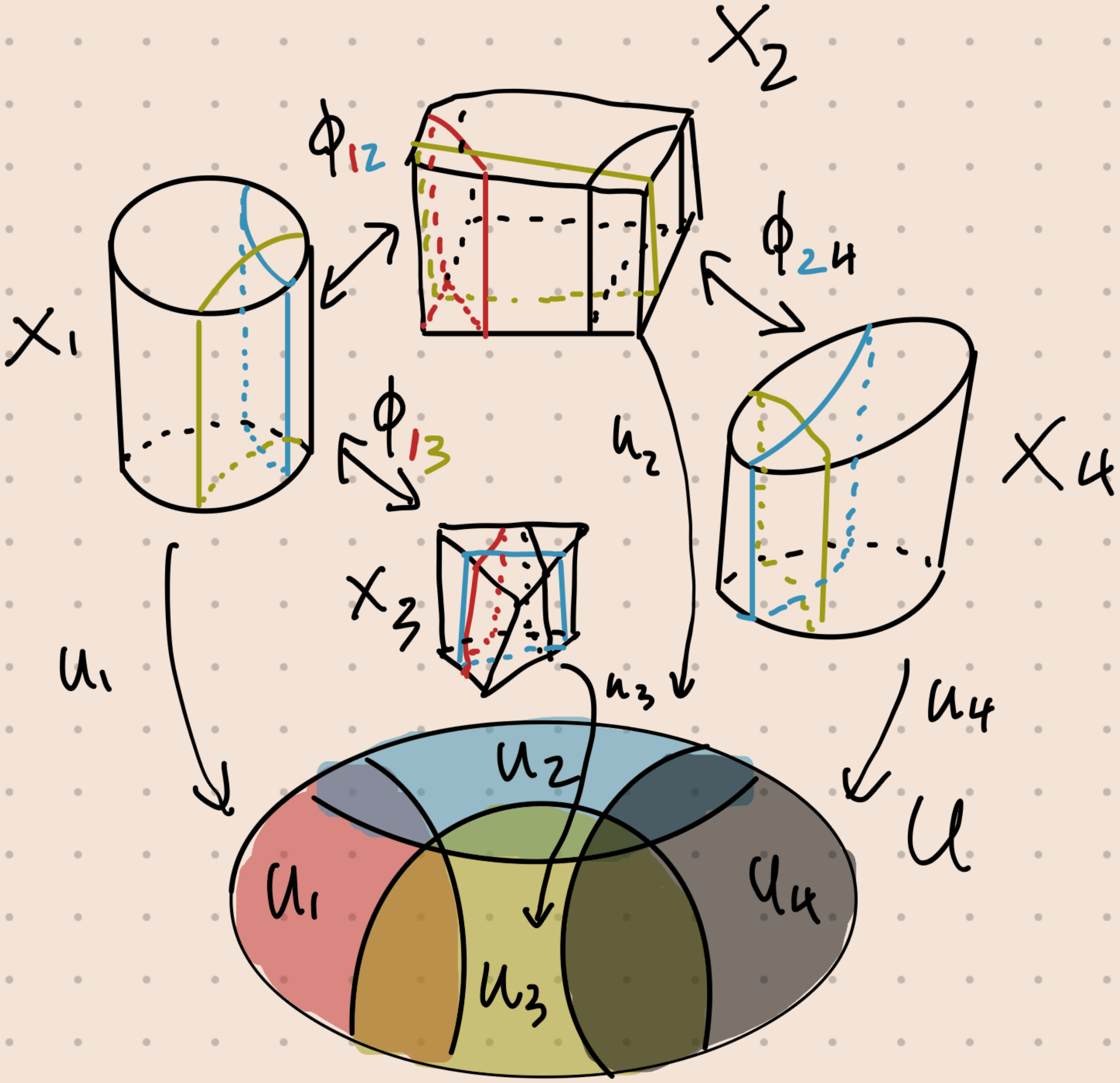
$\forall i$ consider $u_i : X_i \to \mathcal{U}_i$ cont.

$\forall i,j$ $\phi_{ij} : u_j^{-1}\mathcal{U}_{ij} \xrightarrow{\sim} u_i^{-1}\mathcal{U}_{ij}$ homeo. s.t.

$$\begin{tikzcd} u_j^{-1}\mathcal{U}_{ij} \arrow[rr, "\phi_{ij}"] \arrow[dr, "u_j"'] & \arrow[d, phantom, "\Omega"] & u_i^{-1}\mathcal{U}_{ij} \arrow[dl, "u_i"] \\ & \mathcal{U}_{ij} & \end{tikzcd}$$

and $\forall i,j,k$ $\phi_{ik} = \phi_{ij} \circ \phi_{jk}$ over $\mathcal{U}_{ijk}$.

Then $\exists$ $u : X \to \mathcal{U}$ cont. together with $\phi_i : u^{-1}\mathcal{U}_i \xrightarrow{\sim} X_i$
such that $\phi_{ij} = \phi_i \circ \phi_j^{-1}$ $\forall i,j$



this is our archetype for a stack

F fibered over $\mathcal{C}$ $\longleftrightarrow$ Functor (presheaf)

stack $\longleftrightarrow$ Sheaf

[prestack $\longleftrightarrow$ separated presheaf]

For categories fibred in Sets they are the same.

Notation $U_{ij} := U_i \times_U U_j$, $U_{ijk} = U_i \times_U U_j \times_U U_k$
(we are choosing fiber products)

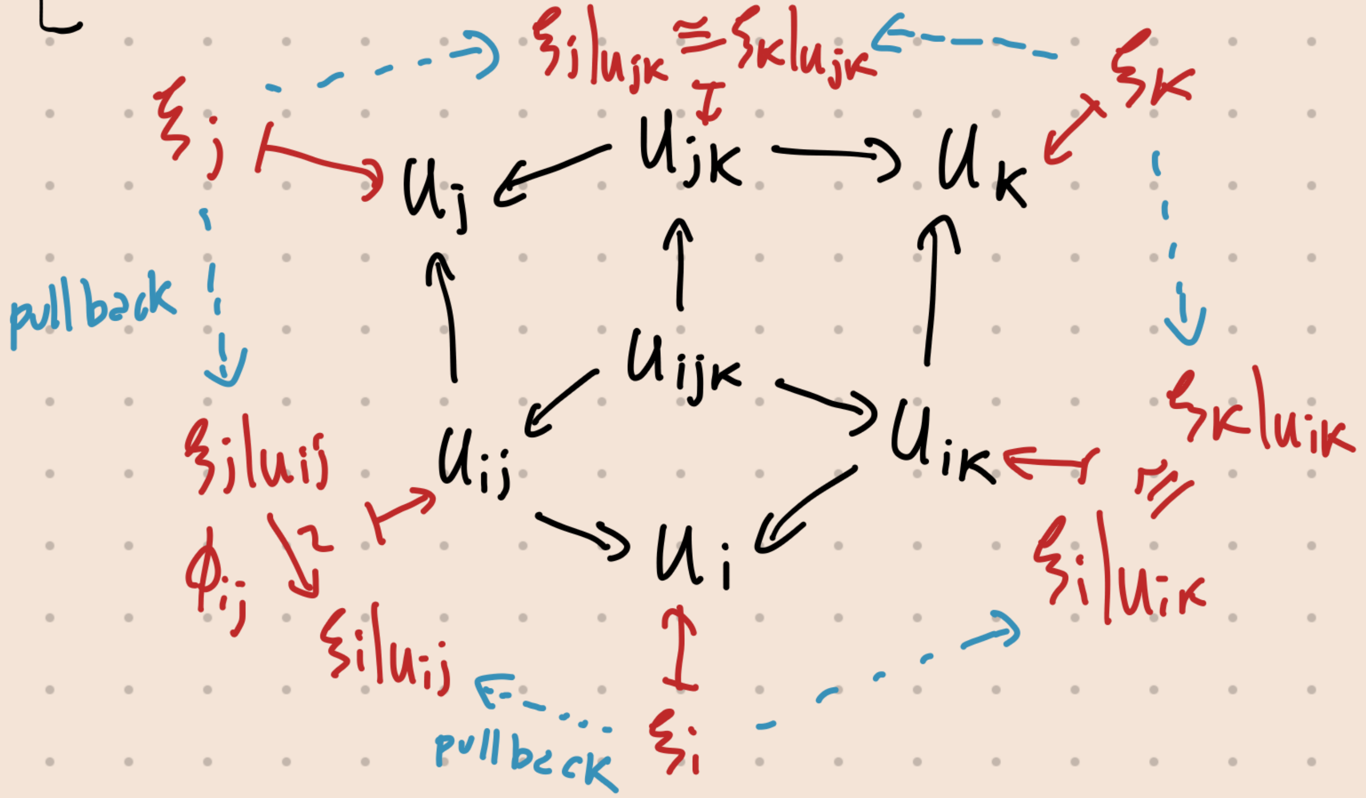
We can directly generalize that data as follows:

Def Let $\mathcal{U} = \{\sigma_i : U_i \to U\} \in \mathrm{Cov}(U)$ for $U \in \mathcal{C}$, $\mathcal{C}$ site.
An object with descent data $(\{\xi_i\}, \{\phi_{ij}\})$ on $\mathcal{U}$ is a collection of objects $\xi_i \in F(U_i)$ together with $\phi_{ij} : pr_2^* \xi_j \xrightarrow{\sim} pr_1^* \xi_i$ in $F(U_{ij})$ s.t. $\forall i,j,k$ $pr_{13}^* \phi_{ik} = pr_{12}^* \phi_{ij} \circ pr_{23}^* \phi_{jk} : pr_3^* \xi_k \to pr_1^* \xi_i$
The ϕ_{ij} are called transition isomorphisms.

[idea: $pr_2^* \xi_j$ "=" $u_j^{-1}(U_i) = u_j^{-1}(U_i \cap U_j)$

$pr_2 : U_{ij} \hookrightarrow U_j$

similarly the other projections are just the appropriate restrictions]



An arrow between objects with descent data

$\{\alpha_i\} : (\{\xi_i\}, \{\phi_{ij}\}) \to (\{\eta_i\}, \{\psi_{ij}\})$

is a collection of $\alpha_i : \xi_i \to \eta_i$ in $F(U_i)$

such that

$$\begin{tikzcd} pr_2^* \xi_j \arrow[r, "pr_2^* \alpha_j"] \arrow[d, "\phi_{ij}"'] \arrow[dr, phantom, "\circlearrowleft"] & pr_2^* \eta_j \arrow[d, "\psi_{ij}"] \\ pr_1^* \xi_i \arrow[r, "pr_1^* \alpha_i"'] & pr_1^* \eta_i \end{tikzcd}$$

// map between the objects which preserves transition isomorphisms //

Objects with descent data on $\mathcal{U}$ form a category, $\mathcal{F}(\mathcal{U}) = \mathcal{F}(\{U_i \to U\})$

Rem $\forall\, \xi \in \mathcal{F}(U)$ we can get some $(\{\xi_i\}, \{\phi_{ij}\}) \in \mathcal{F}(\mathcal{U})$ as follows: $\xi_i = \sigma_i^* \xi$, $\phi_{ij} : pr_2^* \sigma_j^* \xi \xrightarrow{\sim} pr_1^* \sigma_i^* \xi$

is the iso. that comes from the fact that it is cart. over the pullback of ξ to U_{ij}

Similarly we get morphisms and the diagrams commute

We have $\mathcal{F}(U) \to \mathcal{F}(\mathcal{U})$ functor

The construction is idependent on the cleavage up to iso.

Def Data is effective if in essential image

Def Let $\mathcal{U} = \{U_i \to U\} \in Cov(U)$ $(\{\xi_i\}, \{\xi_{ij}\}, \{\xi_{ijk}\})$ With cartesian arrows where appropriate (ex: $\xi_{ijk} \to \xi_{ij}$, $\xi_{ik} \to \xi_k$ etc)

Then define $\mathcal{F}_{desc}(\mathcal{U})$

Rem Choice of cleavage $\rightsquigarrow$ $\mathcal{F}(\mathcal{U}) \to \mathcal{F}_{desc}(\mathcal{U})$

Def Let $U \in \mathcal{C}$, a sieve on U is a subfunctor of $h_U : \mathcal{C}^{op} \to Set$

Rem A cover $\mathcal{U} = \{U_i \to U\}$ defines a sieve $h_{\mathcal{U}}$:

$$h_{\mathcal{U}}(X) = \left\{ X \to U \;\middle|\; \exists i \text{ s.t. } \begin{tikzcd} X \arrow[r] \arrow[d] & U \\ U_i \arrow[ur] & \end{tikzcd} \right\}$$

Prop $Hom_{\mathcal{C}}(h_{\mathcal{U}}, \mathcal{F}) \to \mathcal{F}_{desc}(\mathcal{U})$ is an equivalence

The map sends $\gamma : h_{\mathcal{U}} \to \mathcal{F}$ to

$$\left(\{\gamma_{U_i}(U_i \to U)\}, \{\gamma_{U_{ij}}(U_{ij} \to U)\}, \{\gamma_{U_{ijk}}(U_{ijk} \to U)\} \right)$$

factors throug U_i

The arrows are given again by looking at where γ sends them.

STACKS

Def $\mathcal{F} \to \mathcal{C}$ fibered, $\mathcal{C}$ site, then

$\mathcal{F}$ is prestack over $\mathcal{C}$ if $\forall \{U_i \to U\}$ cover in $\mathcal{C}$

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(\{U_i \to U\}) \text{ fully faithful.}$$

$\mathcal{F}$ is a stack if $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$ eq. of cat

$(\{\xi_i\}, \{\varphi_{ij}\}) \in \mathcal{F}(\{U_i \to U\})$ is effective if it is in the essential image of $\mathcal{F}(U) \to \mathcal{F}(\{U_i \to U\})$

Prop $\mathcal{C}$ site, $F : \mathcal{C}^{op} \to$ Set functor then if we look at it as fibered cat. then
(i) F prestack iff separated presheaf
(ii) F stack iff sheaf

Pf $F(U)$ is what we expect, $F(\{U_i \to U\})$ is data of $(\xi_i) \in \prod F(U_i)$ s.t. $\xi_i|_{U_{ij}} = \xi_j|_{U_{ij}}$

$F(U) \to F(\{U_i \to U\})$ fully faithful $\Leftrightarrow$ injective
equivalence $\Leftrightarrow$ bijective □

Prop $\mathcal{F}$ Fib. over $\mathcal{C}$. $\mathcal{F}$ is prestack iff $\forall S \in \mathcal{C}$ & $\forall \xi, \eta \in \mathcal{F}(S)$, $\underline{\mathrm{Hom}}_S(\xi, \eta) : (\mathcal{C}/S)^{op} \to (\mathrm{Set})$ is a sheaf in the comma topology

Sketch $\mathcal{F}(U) \to \mathcal{F}(\mathcal{U})$ fully faithful means that $\mathrm{Hom}_U(\xi, \eta) = \mathrm{Hom}_{\mathcal{U}}((\{\xi_i\}, \{\alpha_{ij}\}), (\{\eta_i\}, \{\beta_{ij}\}))$ and this means that compatible data (coming from objects) gives uniquely, i.e. $\underline{\mathrm{Hom}}_U(\xi, \eta)$ is a sheaf. □

Cor $Sh/\mathcal{C} \to \mathcal{C}$ is a stack

Sketch prestack by criterion. If $F_i \in Sh U_i$ with ϕ_{ij}

$$F(T) = eq\left(\prod F_i(T \times_U U_i) \rightrightarrows \prod F_i(T \times_U U_{ij})\right)$$

the rest is tedious checking □

Prop. Descent data is functorial in all these ways:

- If $F: \mathcal{F} \to \mathcal{G}$ morphism of fibered categories
  $\forall \mathcal{U}$ there is a functor $\mathcal{F}(\mathcal{U}) \xrightarrow{F} \mathcal{G}(\mathcal{U})$
  $(\{\xi_i\}, \{\phi_{ij}\}) \mapsto (\{F\xi_i\}, \{F\phi_{ij}\})$
  same on morphisms
- If $\mathcal{U} \in Cov(U)$, $V \to U$ then $\mathcal{F}(\{U_i \to U\}) \to \mathcal{F}(\{V \times U_i \to V\})$
  (take fib. prod.)
- If $U \in \mathcal{C}$, $\mathcal{U} \in Cov(U)$ and $\mathcal{V} \in Cov(U)$ a refinement of $\mathcal{U}$
  we get $\mathcal{F}(\mathcal{U}) \to \mathcal{F}(\mathcal{V})$ (take data of the data)

Prop
- If $\mathcal{F} \xrightarrow{F} \mathcal{G}$ equivalence, $\mathcal{F}(U) \to \mathcal{F}(\mathcal{U})$ equiv.
  then $\mathcal{G}(U) \to \mathcal{G}(\mathcal{U})$ equiv.
- being stack/prestack is stable under equivalence

Rem $\mathcal{F} \to \mathcal{C}$ stack iff $\forall \mathcal{U} \in Cov(U)$

$$\mathrm{Hom}_{\mathcal{C}}(h_U, \mathcal{F}) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(h_{\mathcal{U}}, \mathcal{F})$$

is an equivalence