Ref: Vistoli, Notes on Grothendieck topologies, fibered categories, and descent theory, 2.3.1-3
Olsson, Algebraic Spaces and Stacks, 2.1-2

<u>Goal</u> generalize the notion of a sheaf on a top sp

<u>Recall</u> X top sp $\quad Op(X)$ — objects: open subsets of X; arrows: inclusions

presheaf = functor: $Op(X)^{op} \to (Set)$ ← or grps/rings

sheaf = presheaf + gluing conditions

Grothendieck topologies

~~open sets~~ coverings

DEF Let $\mathcal{C}$ be a category. A <u>Grothendieck topology</u> on $\mathcal{C}$ is the assignment to each object U of $\mathcal{C}$ of a collection $Cov(U)$ of sets of arrows $\{U_i \to U\}_{i\in I}$, called <u>coverings</u>, s.t.

(i) $V \to U$ iso $\Rightarrow \{V \to U\} \in Cov(U)$

(ii) $\{U_i \to U\}_i \in Cov(U)$, $V \to U$ arrow in $\mathcal{C}$ $\Rightarrow U_i \times_U V$ exist in $\mathcal{C}$ and $\{U_i \times_U V \xrightarrow{pr_2} V\}_i \in Cov(V)$

(iii) $\{U_i \to U\}_i \in Cov(U)$, $\forall i$ $\{V_{ij} \to U_i\}_j \in Cov(U_i)$
$\Rightarrow \{V_{ij} \to U_i \to U\} \in Cov(U)$

A category with a Gr top is called a <u>site</u>

<u>Rmk</u> actually, in SGA4 ~~what~~ what we have defined is called a pretopology

<u>Examples</u>

• (classical top) $\mathcal{C} = Op(X)$ ← fixed top sp $\quad Cov(U) = \{$open coverings of $U\}$

DEF $\{U_i \xrightarrow{f_i} U\}$ is <u>jointly surjective</u> if $\bigcup f_i(U_i) = U$ ← set-theoretic

• (global class.) $\mathcal{C} = (Top)$ $\quad$ ~~Cov(X)~~ = {jointly surj $\{U_i \to X\}$ } — a covering of X is a
$\uparrow$ open embedding (open continuous, INJ)

• (global Zariski) $\mathcal{C} = (Sch/X)$ fixed scheme

a covering of $U \to X$ is a J. surj $\{U_i \to U\}$

X-morphism, open embedding

morph $V \to W$ that gives iso $V \xrightarrow{\sim}$ open subscheme of W

(Recall)
DEF $f: X \to Y$ morph. of schemes is formally unramified/smooth/étale if $\forall$ affine Y-scheme $Y' \to Y$, $\forall$ closed emb $Y_0' \to Y'$ def. by a nilpotent ideal, the map $Hom_Y(Y', X) \to Hom_Y(Y_0', X)$ is inj/surj/bij

If f is also locally of fin. pres., then f is called unramified/smooth/étale

many equivalent definitions...
smooth + unram, flat + unram...

• ② (small étale) X scheme, $\mathcal{C} = Et(X) :=$ full subcat of (Sch/X) whose objects are $U \to X$ étale

(RmK $\begin{array}{ccc} U & & V \\ & {}_{ét}\searrow \;\; \swarrow_{ét} & \\ & X & \end{array}$ $\Rightarrow U \to V$ étale [PROP. 1.3.4(iii) Olsson])

a cov of U is a J. surj $\{U_i \to U\}$

• ③ (big étale) $\mathcal{C} = (Sch/X)$ a cov. of U is a J. surj $\{U_i \to U\}$
X-morph, étale

• ① (smooth) X scheme, $\mathcal{C} =$ full subcat of (Sch/X) whose objects are $U \to X$ smooth

a cov of U is a J. surj $\{U_i \to U\}$
smooth

Sheaves

DEF $\mathcal{C}$ category. A presheaf on $\mathcal{C}$ is a functor $F: \mathcal{C}^{op} \to (Set)$

DEF $\mathcal{C}$ site, F presheaf on $\mathcal{C}$

• F is separated if $\forall$ object U, $\forall \{U_i \xrightarrow{\varphi_i} U\} \in Cov(U)$, $\forall a, b \in F(U)$

$F(U_i) \ni \varphi_i^* a = \varphi_i^* b \;\; \forall i \Rightarrow a = b$

$[\varphi_i^* = F(\varphi_i)]$

• F is a <u>sheaf</u> if $\forall$ object U, $\forall \{U_i \to U\} \in Cov(U)$

~~given~~ if $\{a_i \in F(U_i)\}_i$ s.t. $pr_1^* a_i = pr_2^* a_j \in F(U_i \times_U U_j)$ $\forall i, j$

~~then~~ then $\exists!\, a \in F(U) : \varphi_i^* a = a_i \ \forall i$

DEF F, G sheaves on $\mathcal{C}$. A <u>morphism of sheaves</u> $F \to G$ is a natural transformation of functors.

<u>Rmks</u> • sheaf $\Rightarrow$ separated

• one can define sheaves of grps, rings...

$F : \mathcal{C}^{op} \to (Gp)/(Ring)$ ... is a sheaf if its composite with the forgetful functor $\to (Set)$ is a sheaf

<u>Thm</u> [2.2.4 Olsson] $\mathcal{C}$ site. The inclusion (sheaves on $\mathcal{C}$) $\hookrightarrow$ (presheaves on $\mathcal{C}$) has a left adjoint $F \mapsto F^a$ ← <u>sheafification</u>

Idea of pf) left adj of: • (separated presheaves on $\mathcal{C}$) $\hookrightarrow$ (presheaves on $\mathcal{C}$)

$F \mapsto (U \mapsto F(U)/\sim)$ ← $a \sim b$ if $\exists \{U_i \to U\} \in Cov(U)$ s.t. a, b have same image under $F(U) \to \prod_{i \in I} F(U_i)$

• (sheaves on $\mathcal{C}$) $\hookrightarrow$ (separated presheaves on $\mathcal{C}$)

$$F \mapsto \Big(U \mapsto \coprod_{\text{pairs}} \big(\underbrace{\{U_i \to U\}}_{\in\, Cov(U)},\ \underbrace{\{a_i\}}_{\in\, Eq\left(\prod F(U_i) \to \prod F(U_i \times_U U_j)\right)}\big) \Big/ \sim \Big)$$

$\square$

fppf & fpqc topologies

<u>Motivation</u>

$$\begin{tikzcd} X \times_Y Y' \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

f has $P \Rightarrow f'$ has P for many properties P

("stable under base change")

e.g. qc-compact, finite type, loc of finite type, affine, [Stacks project ~~REMARK~~ 65.4.1]

closed emb, open emb, emb

when does the converse hold?

i.e. $\{Y_i \to Y\}$ covering, $X \times_Y Y_i$ has P $\forall i \overset{?}{\Rightarrow} X \to Y$ has P

If g has good properties...

Fppf site X sch, $\mathcal{C} = (Sch/X)$ (or $\mathcal{C} = (Sch)$)

a cov. of U is a j. surj. $\{U_i \to U\}$

flat + loc of finite pres

DEF A morphism of schemes $f: X \to Y$ is Fpqc if

- it's flat
- it's surj

] "faithfully flat"

- every qc-comp open subset of Y is the image of a " " " X

] for a surj morphism it's equivalent to... (PROP 2.33 Vistoli)

PROP [2.35 Vistoli]

- being Fpqc is stable under composition and base change
- $f: X \to Y$ morphism, $\{V_i \to Y\}$ open covering s.t. $f|_{f^{-1}(V_i)}: f^{-1}(V_i) \to V_i$ is Fpqc $\forall i \Rightarrow f$ ~~[illegible]~~ is fpqc

Fpqc top $\mathcal{C} = (Sch/X)$ ~~[illegible]~~ a cov of U is $\{U_i \to U\}$ s.t. $\coprod U_i \to U$ is Fpqc

induced by

PROP $\Rightarrow$ it's a Grothendieck top

Fpqc top is finer than] Stacks project Lemma 34.9.7

Fppf "] étale $\Rightarrow$ flat & loc of fin pres

étale "] open embeddings are étale

Zariski

PROP $X \to Y$ morphism of schemes, $\{Y_i \to Y\}$ Fpqc covering.

If $X \times_Y Y_i \to Y_i$ has P $\forall i$, then $X \to Y$ has P

P = qc-comp, separated, surj, loc of finite type/pres, proper, flat, affine, finite, smooth, unram, étale

ecc

[Stacks project Section 35.23]

35.24: fppf

~~P immersion~~

~~P = "being an immersion"~~

also P = "being an immersion"